SYNOPSIS

1. Introduction

The study of partial differential equations has been the cornerstone of applied sciences as they are of utmost importance in fields like physics, biology, chemistry and other applied sciences. But the complex physical phenomena in real life could be best described by nonlinear evolutionary partial differential equations (NLEPDEs). More precisely, the natural world has been revealed as nonlinear due to the rise of nonlinear wave propagation problems [1]. Thus, advancement in science and mathematical modelling dealing with complex physical phenomena give rise to induction of more and more NLEPDEs with constant and variable coefficients as well as their systems. With the increasing number of variables, higher-dimensional nonlinear equations may contain more information and properties with respect to the natural phenomena. In the last twenty years, there is a tremendous growth in mathematical formulation of wide variety of NLEPDEs in order to study about the modeling of nonlinear processes [2]. Infact, it is beyond our imagination to reckon any field of science or mathematics where the impact of NLEPDEs is void.

Now a days, most of scientific problems are solved by understanding the nonlinear phenomena behind it. Such phenomena can be best understood by seeking exact solutions of the corresponding NLEPDEs for concerned problem. Various authors are also imperative to explore the exact solutions due to its vitality in various fields and its adherence to the sense of reality. Exact solutions play very important role for checking validity of numerical and approximation techniques and programs. Thus, it is significantly important to solve NLEPDEs from both theoretical and practical points of view. Realizing the growing importance of exact solutions provoked numerous mathematicians and physicists to develop different variety of methods in order to find such exact solutions.

In existing literature, there are many techniques for solving NLEPDEs such as inverse scattering transform [3], Jacobi elliptic function analysis [4], extended Riccati equation rational

expansion method [5], variational iteration method [6], Hirota method [7], $\left(\frac{G'}{G}\right)$ expansion method [8], homogeneous balance method [9], nonclassical symmetry method [10], first integral method [11], multiple exp-function algorithm [12], mapping deformation method [13], bilinear Bäcklund transformation [14], extended trial method [15], Lie group analysis [16], $\exp(-\phi(\xi))$-expansion method [17], extended tanh-function method [18], Painlevé analysis method [19], $\tan\left(\frac{\phi(\xi)}{2}\right)$-expansion method [20], novel test function method [21], sine-cosine function method [22], modified extended direct algebraic method [23], modified Kudryashov method [24], modified transformed rational function method [25], extended sinh-Gordon equation expansion technique [26], $\left(\frac{G'}{G^2}\right)$-expansion method [27], power series method [28] and many more. The demand for reliable applications encourages deeper understanding of the aforementioned mathematical methods in order to solve more and more NLEPDEs.

The physics behind the NLEPDEs such as Korteweg-de Vries (KdV) equation [29], Schrödinger equation [30], Sawada-Kotera [31], Kaup-Kupershmidt [32], Boussinesq equation [33], Kawahara equation [34], Kadomtsev-Petviashvili (KP) equation [35] etc. agrees well with the characteristics of some waves [36, 37, 38, 39]. Due to presence of nonlinearity and high dimensional spatial variables in the above mentioned NLEPDEs, it is a difficult job for us to determine their exact solutions. This motivated us to determine exact solutions of several NLEPDEs via applying some of the analytic methods which have not been engaged with them in the literature. Such new exact solutions have been explored in this direction and to the best of our knowledge, this work not been reported so far. Also, our solutions consists of many arbitrary constants which may be further useful in progressive research and development.

2. Objectives

The main focus of the proposed work is broadly devoted to obtain the exact solutions of variety of important NLEPDEs or systems of such equations. We took different routes to fulfill our target for finding the exact solutions of NLEPDEs;

1. To directly explore exact solutions of several NLEPDEs by applying various efficient techniques namely; $\left(\frac{G'}{G^2}\right)$-expansion method, $\tan\left(\frac{\phi(\xi)}{2}\right)$-expansion method.
2. To examine classical symmetries, reduce the NLEPDEs to various ordinary differential equations (ODEs) and to further find exact solutions.

3. To investigate integrability properties, obtain bilinear equations and then applying a novel test function method to solve NLEPDEs.

4. To extend a method known as $\exp(-\phi(\xi))$-expansion method by considering negative powers of $\phi(\xi)$. We named the method as "extended $\exp(-\phi(\xi))$-expansion method" and further applied on different NLEPDEs.

3. Organization of the Thesis

The thesis is segmented into eight chapters, in which first chapter is an introductory chapter, chapters two to seven are focused to explore variety of exact solutions of several NLEPDEs. Finally, the conclusion and future scope is provided in the last chapter. An overview of the research work carried out in each chapter of the thesis is as follows:

Chapter 1: Introduction

This chapter imparts a brief review of present work. Importance of exact solutions for NLEPDEs is mentioned. It also presents the methodologies utilized in the thesis and a brief account of the related studies made by various authors in the literature. Besides, research gap and objectives are also discussed.

Chapter 2: Exact Solutions of Whitham-Broer-Kaup Equations

This chapter reveals the exact solutions of a completely integrable model namely; Whitham-Broer-Kaup (WBK) equations [40] via generalized $\left(\frac{G'}{G^2}\right)$-expansion method. The solutions of WBK equations have been explored in terms of exponential functions, trigonometric functions and rational functions. Moreover, arbitrary constants have been selected appropriately in the solutions in order to plot various interesting solitary waves. Also, the innovative as well as distinct exact solutions raised in this chapter may hold powerful applications in various fields of mathematical physics.

Chapter 3: Analytic Solutions of the Coupled Potential KdV Equations

The focus of the current chapter is on the following coupled potential KdV equations [41],

$$\begin{aligned} -u_t + \tfrac{1}{2}u_{xxx} + \tfrac{1}{2}v_{xxx} + 2u_x^2 + v_x^2 = 0, \\ -v_t + \tfrac{1}{2}u_{xxx} + \tfrac{1}{2}v_{xxx} + 2v_x^2 + u_x^2 = 0, \end{aligned} \tag{1}$$

where, u and v are functions of spatial variable x and temporal variable t. Here, we have applied Painlevé analysis on the considered system of equations. Further, classical Lie symmetry analy-

sis has been employed to derive corresponding vector fields, optimal system and similarity variables. Finally, various exact solutions are derived in detail by using ansätz and $\left(\frac{G'}{G^2}\right)$-expansion method. Also, 3D plots of some of the obtained solutions are also explored for future endeavors in physical and applied sciences.

Chapter 4: Application of $\tan\left(\frac{\phi(\xi)}{2}\right)$**-Expansion Method to** $(1+2)$**-Dimensional Nonlinear Schrödinger Equation**

In this chapter, exact solutions of $(1+2)$-dimensional nonlinear Schrödinger equation (NLSE) with dual power law nonlinearity [42] have been determined by applying $\tan\left(\frac{\phi(\xi)}{2}\right)$-expansion method. It is worthy to note that several already published solutions of $(1+2)$-dimensional NLSE can be recovered by taking appropriate values of the involved arbitrary constants. Various explicit physical structures of several types of solitary waves are graphically represented distinctly. The obtained solutions may be useful in validating several numerical schemes and their accuracy.

Chapter 5: Painlevé Analysis, Bilinearization and Exact Solutions of $(1+3)$**-Dimensional BKP-Boussinesq Equation and** $(1+3)$**-Dimensional KP-Boussinesq Equation**

In this chapter, we have considered two equations as follows:

$(1+3)$-dimensional B-type Kadomstev-Petviashvili (BKP)- Boussinesq equation [43],

$$u_{ty} - u_{xxxy} - 3(u_x u_y)_x + u_{tt} + 3u_{xz} = 0, \tag{2}$$

and $(1+3)$-dimensional Kadomstev-Petviashvili (KP)- Boussinesq equation [44],

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} + u_{tt} - u_{zz} = 0, \tag{3}$$

where, $u = u(x,y,z,t)$ and subscripts indicates derivatives of u partially with respect to the corresponding variable. It has been found that the considered equations pass Painlevé test which implies affirmation towards the complete integrability. Additionally, bilinear equations of the considered equations has been constructed by truncating the Painlevé expansions at constant level and using the Bell polynomial approach. Then abundant exact solutions involving various arbitrary constants have been obtained in uniform manner by using novel test function. The acquired solutions have also been plotted graphically depicting distinct types of solitary waves.

Chapter 6: Analytic Study of $(1+2)$**-Dimensional Generalized Hirota-Satsuma-Ito Equa-**

tion and $(1+3)$-Dimensional Generalized Kadomstev-Petviashvili Equation

For the present chapter, we have emphasized on obtaining new analytic solutions of the $(1+2)$-dimensional Hirota-Satsuma-Ito equation with general coefficients (gHSI) [45] and $(1+3)$-dimensional generalized Kadomstev-Petviashvili equation (gKP) [46]. Based on the derived bilinear form of gHSI equation, a novel test function has been appointed to formally derive various exact solutions containing abundant arbitrary constants. On the other hand, it has been shown that gKP equation does not pass Painlevé test which implies failure of integrability. But, we have still truncated Painlevé expansion in order to determine the bilinear equation of gKP equation via Bell polynomial approach. Then, various exact solutions involving various arbitrary constants are furnished in uniform manner by using novel test function. New solutions consist of hyperbolic, trigonometric and exponential functions. 3-dimensional plots of some of the exact solutions of both the considered equations, determined in this research have also been provided in uniform manner.

Chapter 7: Exact Solutions of $(1+3)$-Dimensional Generalized Breaking Soliton Equation and $(1+4)$-Dimensional Fokas Equation

An extension of the celebrated KdV equation known as $(1+3)$-dimensional generalized breaking soliton equation [47] has been considered in this chapter,

$$v_{xt}+p_1v_x(v_{xy}+v_{xz})+p_2v_{xx}(v_y+v_z)+p_3(v_{xxxy}+v_{xxxz})=0, \tag{4}$$

where, $p_1,\ p_2$ and p_3 are nonzero constants.

Another novel equation containing four spatial variables has been considered in this chapter; namely, $(1+4)$-dimensional Fokas equation [48]

$$v_{xt}-\frac{1}{4}v_{xxxp}+\frac{1}{4}v_{xppp}+\frac{3}{2}(v^2)_{xp}-\frac{3}{2}v_{yz}=0, \tag{5}$$

where, $v=v(x,y,z,t,p)$. Eq. (5) is a generalization of the Davey-Stewartson (DS) and KP equations in three and four spatial dimensions.

Exact solutions for Eqs. (4) and (5) have been scrutinized by incorporating an effective approach namely; extended $\exp(-\phi(\xi))$-expansion method. Rational, hyperbolic function, trigonometric function solutions with arbitrary constants are determined by applying the above mentioned method. The obtained exact solutions are completely new. It is noteworthy that the extended

$\exp(-\phi(\xi))$-expansion method can be successfully applied to a large variety of NLEPDEs in order to explore their exact solutions.

Chapter 8: Conclusions and Future Scope

Summary of the work has been provided in this final chapter. Also, some future scope has been given as the novelty in development of different variety of exact solutions might help the researchers to do detailed study about the concerned nonlinear phenomena.

REFERENCES

[1] Ablowitz, M.J., Clarkson, P.A., *"Solitons, nonlinear evolution equations and inverse scattering"*, Cambridge University Press, vol. 149, Dec. 1991.

[2] Leung, A.W., *"Nonlinear systems of partial differential equations: Applications To Life And Physical Sciences"*, World Scientific Publishing Company, Aug. 2009.

[3] Vakhnenko, V.O., Parkes, E.J., Morrison A.J., *"A Bäcklund transformation and the inverse scattering transform method for the generalised Vakhnenko equation"*, Chaos, Soliton & Fractals, vol. 17, pp. 683-692, Aug. 2003.

[4] Dai, C.Q., Zhang, J.F., *"Jacobian elliptic function method for nonlinear differential-difference equations"*, Chaos, Soliton & Fractals, vol. 27, pp. 1042-1049, Feb. 2006.

[5] Song, L.N., Wang, Q., Zheng, Y., Zhang, H.Q., *"A new extended Riccati equation rational expansion method and its application"*, Chaos, Soliton & Fractals, vol. 31, pp. 548-556, Feb. 2007.

[6] He, J.H., *"Variational iteration method-some recent results and new interpretations"*, Journal of Computational and Applied Mathematics, vol. 207, pp. 3-17, Oct. 2007.

[7] Wazwaz, A.M., *"The Hirota's direct method for multiple-soliton solutions for three model equations of shallow water waves"*, Applied Mathematics and Computation, vol. 201, pp. 489-503, Jul. 2008.

[8] Wang, M., Li, X., Zhang, J., *"The* $\left(\frac{G'}{G}\right)$*-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics"*, Physics Letters A, vol. 372, pp. 417-423, Jan. 2008.

[9] Rady, A.S.A., Osman, E.S., Khalfallah, M., *"The homogeneous balance method and its application to the Benjamin-Bona-Mahoney (BBM) equation"*, Applied Mathematics and Computation, vol. 217, pp. 1385-1390, Oct. 2010.

[10] El-Sabbagh, M.F., Ali, A.T., *"Nonclassical symmetries for nonlinear partial differential equations via compatibility"*, Communications in Theoretical Physics, vol. 56, pp. 611-616, Oct. 2011.

[11] Hosseini, K., Ansari, R., Gholamin, P., *"Exact solutions of some nonlinear systems of partial differential equations by using the first integral method"*, Journal of Mathematical Analysis and Applications, vol. 387, pp. 807-814, Mar. 2012.

[12] Ma, W.X., Zhu, Z., *"Solving the $(3+1)$-dimensional generalized KP and BKP equations by the multiple exp-function algorithm"*, Applied Mathematics and Computation, vol. 218, pp. 11871-11879, Aug. 2012.

[13] Hong, B., Lu, D., *"New Jacobi elliptic function-like solutions for the general KdV equation with variable coefficients"*, Mathematical and Computer Modelling, vol. 55, pp. 1594-1600, Feb. 2012.

[14] Ma, W.X., Abdeljabbar, A., *"A bilinear Bäcklund transformation of a $(3+1)$-dimensional generalized KP equation"*, Applied Mathematics Letters, vol. 25, pp. 1500-1504, Oct. 2012.

[15] Gurefe, Y., Misirli, E., Sonmezoglu, A., Ekici, M., *"Extended trial equation method to generalized nonlinear partial differential equations"*, Applied Mathematics and Computation, vol. 219, pp. 5253-5260, Jan. 2013.

[16] Kaur, L., Gupta, R.K., *"On symmetries and exact solutions of the Einstein-Maxwell field equations via the symmetry approach"*, Physica Scripta, vol. 87, pp. 035003, Feb. 2013.

[17] Hafez, M.G., Alam, M.N., Akbar, M.A., *"Traveling wave solutions for some important coupled nonlinear physical models via the coupled Higgs equation and the Maccari system"*, Journal of King Saud University-Science, vol. 27, pp. 105-112, Apr. 2015.

[18] Zahran, E.H.M., Khater, M.M.A., *"Modified extended tanh-function method and its applications to the Bogoyavlenskii equation"*, Applied Mathematical Modelling, vol. 40, pp. 1769-1775, Feb. 2016.

[19] Wazwaz A.M., Xu, G., *"An extended modified KdV equation and its Painlevé integrability"*, Nonlinear Dynamics, vol. 86, pp. 1455-1460, Nov. 2016.

[20] Manafian, J., Lakestani, M., *"Abundant soliton solutions for the Kundu-Eckhaus equation via* $\tan\left(\frac{\phi(\xi)}{2}\right)$*-expansion method"*, Optik, vol. 127, pp. 5543-5551, Jul. 2016.

[21] Singh, M., Gupta, R.K., *"Exact solutions for nonlinear evolution equations using novel test function"*, Nonlinear Dynamics, vol. 86, pp. 1171-1182, Oct. 2016.

[22] Raslan, K.R., Danaf, T.S.E., Ali, K.K., *"New exact solution of coupled general equal width wave equation using sine-cosine function method"*, Journal of the Egyptian Mathematical Society, vol. 25, pp. 350-354, Jul. 2017.

[23] Arshad, M., Seadawy, A.R., Lu, D., *"Optical soliton solutions of the generalized higher-order nonlinear Schrödinger equations and their applications"*, Optical and Quantum Electronics, vol. 49, pp. 421, Nov. 2017.

[24] Hosseini, K., Kumar, D., Kaplan, M., Bejarbaneh, E.Y., *"New exact traveling wave solutions of the unstable nonlinear Schrödinger equations"*, Communications in Theoretical Physics, vol. 68, pp. 761-767, Dec. 2017.

[25] Sun, Y.L., Ma, W.X., Yu, J.P., Khalique, C.M., *"Exact solutions of the Rosenau-Hyman equation, coupled KdV system and Burgers-Huxley equation using modified transformed rational function method"*, Modern Physics Letters B, vol. 32, pp. 1850282, Aug. 2018.

[26] Bulut, H., Sulaiman, T.A., Baskonus, H.M., *"Dark, bright and other soliton solutions to the Heisenberg ferromagnetic spin chain equation"*, Superlattices and Microstructures, vol. 123, pp. 12-19, Nov. 2018.

[27] Sirisubtawee, S., Koonprasert, S., *"Exact traveling wave solutions of certain nonlinear partial differential equations using the* $\left(\frac{G'}{G^2}\right)$*-expansion method"*, Advances in Mathematical Physics, vol. 2018, pp. 7628651, Jun. 2018.

[28] Arora, R., Chauhan, A., *"Lie Symmetry analysis and some exact solutions of* $(2+1)$*-dimensional KdV-Burgers equation"*, International Journal of Applied and Computational Mathematics, vol. 5, pp. 15, Feb. 2019.

[29] Miura, R.M., *"Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation"*, Journal of Mathematical Physics, vol. 9, pp. 1202-1204, Aug. 1968.

[30] Zakharov, V.E., Shabat, A.B., *"Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in non linear media"*, Journal of Experimental and Theoretical Physics, vol. 34, pp. 62-69, Jan. 1972.

[31] Sawada, K., Kotera, T., *"A method for finding N-soliton solutions for the KdV equation and KdV-like equation"*, Progress of Theoretical Physics, vol. 51, pp. 1355-1367, May 1974.

[32] Kupershmidt, B.A., *"A super KdV equation: an integrable system"*, Physics Letters A, vol. 102, pp. 213-215, May 1984.

[33] Clarkson, P.A., Kruskal, M.D., *"New similarity reductions of the Boussinesq equation"*, Journal of Mathematical Physics, vol. 30, pp. 2201-2213, Oct. 1989.

[34] Wazwaz, A.M., *"New solitary wave solutions to the Kuramoto-Sivashinsky and the Kawahara equations"*, Applied Mathematics and Computation, vol. 182, pp. 1642-1650, Nov. 2006.

[35] Ma W.X., *"Lump solutions to the Kadomtsev-Petviashvili equation"*, Physics Letters A, vol. 379, pp. 1975-1978, Sep. 2015.

[36] Washimi, H., Taniuti, T., *"Propagation of ion-acoustic solitary waves of small amplitude"*, Physical Review Letters, vol. 17, pp. 996-998, Nov. 1966.

[37] Trulsen, K., Dysthe, K.B., *"A modified nonlinear Schrdinger equation for broader bandwidth gravity waves on deep water"*, Wave Motion, vol. 24, pp. 281-289, Nov. 1996.

[38] Vasconcellos, C.F., da Silva, P.N., *"Stabilization of the Kawahara equation with localized damping"*, ESAIM: Control, Optimisation and Calculus of Variations, vol. 17, pp. 102-116, Jan. 2011.

[39] Taylor, L.S., *"The mechanisms of athermal microwave biological effects"*, Bioelectromagnetics, vol. 2, pp. 259-267, May 1981.

[40] Zhang, C., Tian, B., Meng, X.H., Lü, X., Cai, K.J., Geng, T., *"Painlevé integrability and N-soliton solution for the Whitham-Broer-Kaup shallow water model using symbolic computation"*, Zeitschrift für Naturforschung A, vol. 63, pp. 253-260, Jun. 2008.

[41] Foursov, M.V., *"Classification of certain integrable coupled potential KdV and modified KdV-type equations"*, Journal of Mathematical Physics, vol. 41, pp. 6173-6185, Sep. 2000.

[42] Bulut, H., Pandir, Y., Demiray, S.T., *"Exact solutions of nonlinear Schrödinger's equation with dual power-law nonlinearity by extended trial equation method"*, Waves in Random and Complex Media, vol. 24, pp. 439-451, Oct. 2014.

[43] Ma, W.X., Zhu, Z., *"Solving the* $(3+1)$*-dimensional generalized KP and BKP equations by the multiple exp-function algorithm"*, Applied Mathematics and Computation, vol. 218, pp. 11871-11879, Aug. 2012.

[44] Wazwaz, A.M., Tantawy, S.A.E., *"Solving the* $(3+1)$*-dimensional KP-Boussinesq and BKP-Boussinesq equations by the simplified Hirota's method"*, Nonlinear Dynamics, vol. 88, pp. 3017-3021, Jun. 2017.

[45] Ma, W.X., Li, J., Khalique, C.M., *"A study on lump solutions to a generalized Hirota-Satsuma-Ito equation in* $(2+1)$*-dimensions"*, Complexity, vol. 2018, pp. 9059858, Dec. 2018.

[46] Wazwaz, A.M., Tantawy, S.A.E., *"A new* $(3+1)$*-dimensional generalized Kadomtsev-Petviashvili equation"*, Nonlinear Dynamics, vol. 84, pp. 1107-1112, Apr. 2016.

[47] Tariq, K.U.H., Seadawy, A.R., *"Bistable Bright-Dark solitary wave solutions of the* $(3+1)$*-dimensional Breaking soliton, Boussinesq equation with dual dispersion and modified Korteweg-de Vries-Kadomtsev-Petviashvili equations and their applications"*, Results in Physics, vol. 7, pp. 1143-1149, Jan. 2017.

[48] Zhang, S., Tian, C., Qian, W.Y., *"Bilinearization and new multisoliton solutions for the* $(4+1)$*-dimensional Fokas equation"*, Pramana, vol. 86, pp. 1259-1267, 2016.